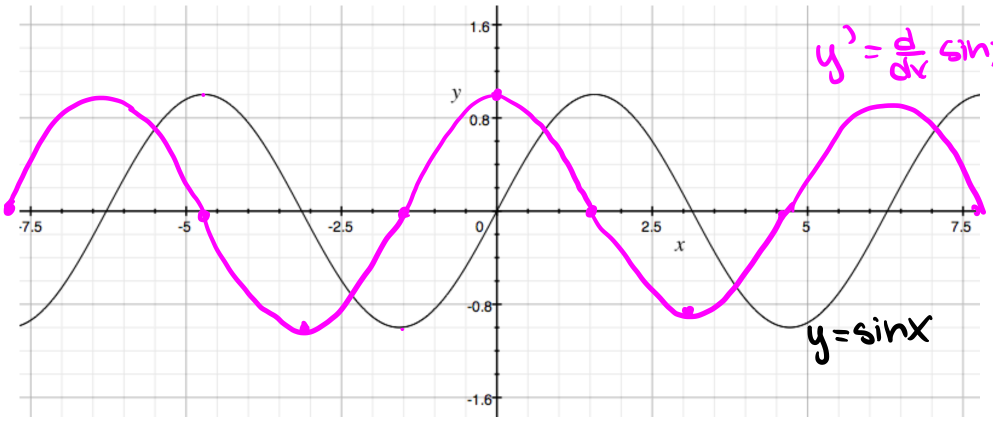
Lecture: 3-3 Derivatives of Trigonometric Functions

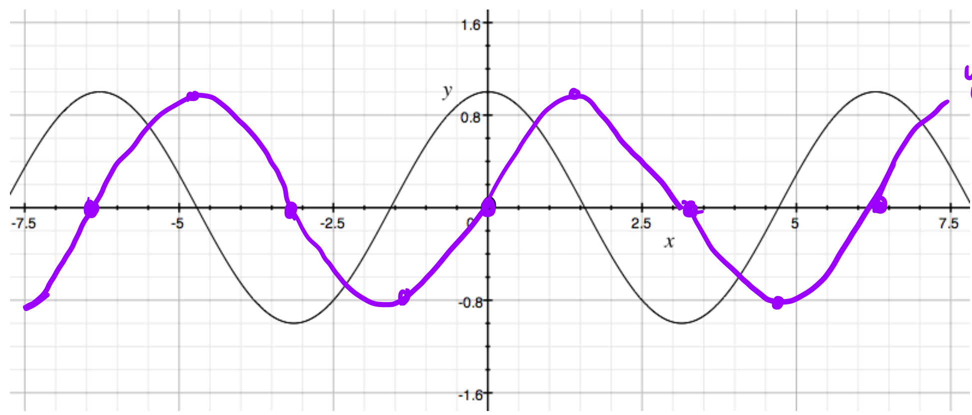
Example 1: Use the graph of $y = \sin x$ to sketch a graph of y'. Guess what y' is.



$y' = \frac{d}{dx}\sin x = \boxed{\cos x}$

appears to be!

Example 2: Use the graph of $y = \cos x$ to sketch a graph of y'. Guess what y' is.



$y' = \frac{d}{dx}\cos x = \boxed{-\sin x}$

$y = \cos x$

Example 3: Using the derivative of $\sin x$ and $\cos x$ find derivatives of:

(a) $y = \tan x = \frac{\sin x}{\cos x}$

now use the quotient rule!

$$y' = \frac{\cos x\left(\frac{d}{dx}\sin x\right) - \sin x\left(\frac{d}{dx}\cos x\right)}{(\cos x)^2}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

(b) $y = \csc x = \frac{1}{\sin x}$

$$y' = \frac{\sin x\left(\frac{d}{dx}1\right) - 1\,\frac{d}{dx}\sin x}{(\sin x)^2}$$

$$= \frac{\sin x\,(0) - \cos x}{\sin^2 x}$$

remember, the derivative of 1 is zero.

$$= \frac{-\cos x}{\sin^2 x} = \frac{-\cos x}{\sin x \cdot \sin x}$$

$$= -\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x} = \boxed{-\cot x \csc x}$$

Note: all the "co" functions have negative derivatives!

Derivatives of Trigonometric Functions: (they're all starred below!)

- $\frac{d}{dx}(\sin x) =$ $\cos x$
- $\frac{d}{dx}(\cos x) =$ $-\sin x$
- $\frac{d}{dx}(\tan x) =$ $\sec^2 x$
- $\frac{d}{dx}(\csc x) =$ $-\csc x \cot x$
- $\frac{d}{dx}(\sec x) =$ $\sec x \tan x$
- $\frac{d}{dx}(\cot x) =$ $-\csc^2 x$

you prove these the same way we did tangent and cosecant.

Example 4: Find the second derivatives of the following functions: ← product rule!

(a) $g(t) = 4\sec t + \tan t$.

$$\boxed{g'(t) = 4\sec t \tan t + \sec^2 t}$$

$$\boxed{g'(t) = \sec t\,(4\tan t + \sec t)}$$

(b) $y = x^2 \sin x$.

$$\boxed{y' = 2x\sin x + x^2\cos x}$$

$$\boxed{y' = x\,(2\sin x + x\cos x)}$$

Example 5: Find an equation of the tangent line to the curve $y = \dfrac{1}{\sin x + \cos x}$ at the point $(0, 1)$.

$$y' = \frac{(\sin x + \cos x)(0) - 1(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{-\cos x + \sin x}{(\sin x + \cos x)^2}$$

$$m = \frac{-\cos 0 + \sin 0}{(\sin 0 + \cos 0)^2} = -1$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -1(x - 0)$$

$$\boxed{y = -x + 1}$$

Example 6: For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

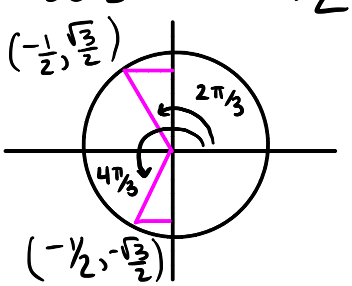
↳ determine when $f'(x) = 0$

$$f'(x) = 1 + 2\cos x$$

$$0 = 1 + 2\cos x$$

$$-1 = 2\cos x$$

$$\cos x = -1/2$$



at $\boxed{x = 2\pi/3 + 2\pi n}$

at $\boxed{x = 4\pi/3 + 2\pi n}$

Example 7: Differentiate $f(x) = \dfrac{\sec x}{1 - \tan x}$ and determine where the tangent line is horizontal.

$$f'(x) = \frac{(1-\tan x)\cdot \sec x \tan x - \sec x\,(-\sec^2 x)}{(1-\tan x)^2}$$

$$= \frac{\sec x\,(\tan x - \tan^2 x + \sec^2 x)}{(1-\tan x)^2}$$

$$= \frac{\sec x\,(\tan x + 1)}{(1-\tan x)^2}$$

trig identity w/ $\sec^2 x$ & $\tan^2 x$...

$(\sin^2 x + \cos^2 x = 1) \div \cos^2 x$

$\tan^2 x + 1 = \sec^2 x$

$1 = \sec^2 x - \tan^2 x$

tangent is horizontal when

$\sec x\,(\tan x + 1) = 0$

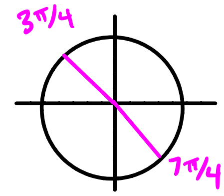
$\sec x = 0$ ← this never happens

$1/\cos x = 0$

$1 = 0$ ← see! crazy!

$\tan x + 1 = 0$

$\tan x = -1$



$\boxed{x = 3\pi/4 + n\pi}$

Generalized Product Rule: How does the product rule genearlize to more than two functions? For example, what is the derivative of $y = f(x)g(x)h(x)$?

$$y' = \frac{d}{dx}\big(f(x)g(x)\big)\cdot h(x) + f(x)g(x)h'(x)$$

$$= \big(f'(x)g(x) + f(x)g'(x)\big)h(x) + f(x)g(x)h'(x)$$

$$= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

Example 8: Differentiate $y = x^2 \tan x \sec x$.

$$y' = \left(\frac{d}{dx}x^2\right)\tan x \sec x + x^2\left(\frac{d}{dx}\tan x\right)\sec x + x^2 \tan x\left(\frac{d}{dx}\sec x\right)$$

$$y' = 2x \tan x \sec x + x^2 \sec^2 x \sec x + x^2 \tan x \sec x \tan x$$

$$\boxed{y' = 2x \tan x \sec x + x^2 \sec^3 x + x^2 \sec x \tan^2 x}$$

Example 9: Find the 51st derivative of $f(x) = \sin x$. Specifically, find the first four or five derivatives and look for a pattern.

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f''''(x) = \sin x$

the pattern cycles in 4 derivatives

If we do $51 \div 4$ we get:

$$4\overline{)51} = 12 \text{ R } 3$$

-40, 11, -8, 3

12 ← 12 full cycles

3 ← then 3 more, so

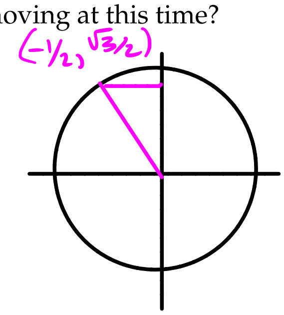
$$\boxed{f^{(51)}(x) = -\cos x}$$

Example 10: A mass on a spring vibrates horizontally on a smooth level surface. Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x is in centimeters.

(a) Find the velocity at time t.

$$\boxed{v(t) = x'(t) = 8 \cos t}$$

(b) Find the position and velocity of the mass at time $t = 2\pi/3$. In what direction is it moving at this time?



position: $x(2\pi/3) = 8 \sin(2\pi/3)$

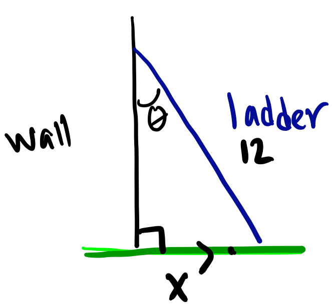
$= 8(\sqrt{3}/2) = \boxed{4\sqrt{3} \text{ cm}}$

velocity $v(2\pi/3) = 8 \cos(2\pi/3)$

$= 8(-1/2) = \boxed{-4 \text{ cm/sec}}$

The spring is going backwards (to the left) as the velocity is negative.

Example 11: A ladder 12 feet long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall how fast does x change with respect to θ when $\theta = \frac{\pi}{6}$?



$\sin\theta = \frac{x}{12} \Rightarrow x = 12 \sin\theta$

$\frac{dx}{d\theta} = 12 \cos\theta$

$= 12 \cos(\pi/6)$

$= 12(\sqrt{3}/2)$

$= 6\sqrt{3}$

≈ 10.392 ft/rad